LOCALLY COMPACT SIMPLE RINGS HAVING MINIMAL LEFT IDEALS

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Let A be an indiscrete locally compact primitive ring having minimal left ideals. Algebraically, A may be regarded as a dense ring of linear operators containing nonzero linear operators of finite rank on a vector space E over a division ring K, and furthermore if A is simple, then A contains only linear operators of finite rank. The topology of A induces topologies on E and K in a natural way, so that K becomes a locally compact division ring and E a locally compact vector space over K. Theorems about locally compact division rings and locally compact vector spaces imply that if K is indiscrete, then E is necessarily finite dimensional over K, and thus A is the ring of all linear operators on a finite-dimensional vector space over a locally compact division ring. Kaplansky proved that if A has characteristic zero, then K is indeed indiscrete, but he showed by an example that E could be infinite dimensional. Kaplansky remarked [9, p. 459], however, that if A is simple, it seemed unlikely that E could be infinite dimensional, since otherwise "completeness appears to require the presence of linear transformations with infinite-dimensional range." Our principal result in §1 is the verification of this conjecture; however, our argument is based not on the fact that a locally compact ring is complete, but rather on the fact that a locally compact space is a Baire space and on the symmetry between minimal left and right ideals in a simple ring. In §2 we shall present certain results about locally compact vector spaces over discrete division rings; these enable us to prove that a locally compact, indiscrete, central primitive algebra having minimal left ideals is finite dimensional provided its scalar field either is indiscrete, has characteristic zero, or is uncountably infinite. However, there is a very natural example of a locally compact, indiscrete, infinite-dimensional central primitive algebra having minimal left ideals over a discrete countably infinite field of prime characteristic.

1. **Locally compact simple rings having minimal left ideals.** We begin with a summary of the structure theory of a primitive ring A having minimal left ideals. Let e be a minimal idempotent of A, that is, an idempotent such that Ae is a minimal left ideal, or equivalently, such that eA is a minimal right ideal. Then eAe is a division ring, the additive group Ae becomes a right eAe-vector space by defining scalar multiplication to be the restriction of multiplication on A to

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$Ae \times eAe$, and similarly eA becomes a left eAe-vector space by defining scalar multiplication to be the restriction of multiplication on A to $eAe \times eA$. For each $a \in A$, let a_L be the linear operator on the right eAe-vector space Ae defined by $a_L(x) = ax$, and let a_R be the linear operator on the left eAe-vector space eA defined by $a_R(x) = xa$. Then $a \to a_L$ is an isomorphism from A onto a dense ring A_L of linear operators on Ae, $a \to a_R$ is an anti-isomorphism from A onto a dense ring A_R of linear operators on eA, and both A_L and A_R contain nonzero linear operators of finite rank [7, Chapters 2 and 4].

Let us assume further that A is a locally compact ring. If $\mathscr{V}$ is a fundamental system of neighborhoods of zero in A, then $\{Ve : V \in \mathscr{V}\}$ is a fundamental system of neighborhoods of zero in Ae. Indeed, if $V \in \mathscr{V}$, then Ve is a neighborhood of zero in Ae since $V \cap Ae \subseteq Ve$; conversely, if U is a neighborhood of zero in A, then there is a neighborhood W of zero in A such that $We \subseteq U \cap Ae$ as $x \to xe$ is continuous. Similarly, $\{eV : V \in \mathscr{V}\}$ and $\{eVe : V \in \mathscr{V}\}$ are fundamental systems of neighborhoods of zero in eA and eAe respectively. If $V \in \mathscr{V}$ is compact, then Ve, eV, and eVe are also compact since they are continuous images of V; hence Ae, eA, and eAe are all locally compact. A theorem of Otobe [10, Theorem 3], generalized in several ways by Kaplansky [9, Theorems 7–9], implies that $x \to x^{-1}$ is continuous on the set of nonzero elements of eAe, and hence eAe is a locally compact division ring; this also follows from a theorem of R. Ellis [4, p. 78, Exercise 25] applied to the multiplicative group of nonzero elements of eAe. Since the scalar multiplications of the vector spaces Ae and eA are simply restrictions of multiplication on A, they are locally compact topological vector spaces over eAe. The topology of eAe is defined by an absolute value [8, Theorem 8], and hence if eAe is indiscrete, then Ae and eA are necessarily finite dimensional over eAe [1, p. 29, Theorem 3]. As mentioned earlier, Kaplansky [9, Theorem 14] proved that eAe is indeed indiscrete if A is indiscrete and has characteristic zero, and this follows also if A is connected, for then eAe is also connected and hence indiscrete as it is a continuous image of A. From Pontrjagin's theorem on connected locally compact division rings, we therefore obtain the following result:

THEOREM 1. *If A is a connected locally compact primitive ring having minimal left ideals, then A is isomorphic to the ring of all linear operators on a finite-dimensional vector space over either the field of real numbers, the field of complex numbers, or the division ring of quaternions.*

In general, for each $a \in A$ the linear operator a_L on Ae and the linear operator a_R on eA are continuous as multiplication is continuous on A. We topologize A_L and A_R so that the bijections $a \to a_L$ and $a \to a_R$ are homeomorphisms. Both A_L and A_R are then locally compact rings of continuous linear operators, and since multiplication on A is continuous, $(u, x) \to u(x)$ is continuous from $A_L \times Ae$ into Ae and also from $A_R \times eA$ into eA. In particular, for each $x \in Ae$ [respectively, $x \in eA$], $u \to u(x)$ is continuous from A_L into Ae [respectively, from A_R into eA].

The vector spaces Ae and eA possess an important property defined as follows:

DEFINITION. A topological vector space E over a topological division ring K is *straight* if for every nonzero vector $x \in E$, the function $\lambda \to \lambda x$ is a homeomorphism from K onto the one-dimensional subspace of E generated by x.

A Hausdorff vector space over an indiscrete topological division ring whose topology is given by an absolute value is straight [1, p. 25, Proposition 2], and any Hausdorff vector space over a (discrete) finite field is clearly straight.

LEMMA 1. *If A is a locally compact primitive ring having minimal left ideals and if e is a minimal idempotent of A, then the eAe-vector spaces Ae and eA are straight.*

Proof. Let $K=eAe$. The function $\lambda \to e\lambda$ from K onto the subspace $e.K$ of Ae is simply the identity mapping of K and hence is a homeomorphism. Let x be a nonzero vector of Ae. The set of all continuous linear operators on Ae is dense since it contains A_L. In particular, there exist continuous linear operators u and v on Ae such that $u(e)=x$ and $v(x)=e$. Thus $e\lambda \to x\lambda$ is a homeomorphism from $e.K$ onto $x.K$, for it is the restriction of u to $e.K$, and its inverse is the restriction of v to $x.K$. Hence $\lambda \to x\lambda$ is also a homeomorphism from K onto $x.K$. Similarly, the left K-vector space eA is straight.

LEMMA 2. *Let E be a discrete topological vector space over a discrete topological division ring K, and let A be a set of linear operators on E, topologized so that $u \to u(x)$ is continuous from A into E for each $x \in E$. For every $n \geqq 0$, the set F_n of all linear operators in A of rank $\leqq n$ is closed in A.*

Proof. Let w belong to the closure of F_n, and let $x_1, \ldots, x_{n+1}$ be a sequence of $n+1$ vectors. There is a filter $\mathscr{F}$ on F_n converging to w, and hence by hypothesis $\mathscr{F}(x_i) \to w(x_i)$ for $1 \leqq i \leqq n+1$. Since E is discrete, there exists $H_i \in \mathscr{F}$ such that $u(x_i)=w(x_i)$ for all $u \in H_i$. Let $u \in H_1 \cap \cdots \cap H_{n+1}$. As $u \in F_n$, there exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ not all of which are zero such that

$$\lambda_1 u(x_1)+\cdots+\lambda_{n+1}u(x_{n+1}) = 0.$$

Hence

$$\sum_{i=1}^{n+1} \lambda_i w(x_i) = \sum_{i=1}^{n+1} \lambda_i u(x_i) = 0.$$

Therefore rank $w \leqq n$.

LEMMA 3. *Let E be a discrete topological vector space, and let A be a dense ring of linear operators of finite rank on E, topologized so that A is a topological ring and $u \to u(x)$ is continuous from A into E for each $x \in E$. If in addition A is a Baire space and the open additive subgroups of A form a fundamental system of neighborhoods of zero, then A is discrete.*

Proof. If E is finite dimensional, then A is discrete, for if $\{c_1, \ldots, c_s\}$ is a basis of E, then

$$\{0\} = \{u \in A : u(c_i) = 0, 1 \leqq i \leqq s\}$$

is a neighborhood of zero since E is discrete. Therefore we shall assume that E is infinite dimensional.

For each $n \geqq 1$ let F_n be the set of all linear operators in A of rank $\leqq n$. Then $\bigcup_{n \geqq 1} F_n = A$, so by Lemma 2 and our hypotheses there exist $n \geqq 1$, a linear operator $v \in A$, and an open additive subgroup G of A such that $v + G \subseteq F_n$. For any $w \in G$,

$$\text{rank } w \leqq \text{rank } (v+w) + \text{rank } (-v) \leqq n + \text{rank } v,$$

so the ranks of members of G are bounded. Let m be the largest of the ranks of members of G, and let $u \in G$ have rank m. Let $x_1, \ldots, x_m \in E$ be such that $\{u(x_1), \ldots, u(x_m)\}$ is a basis of the range M of u. As E is discrete,

$$V = \{v \in G : v(x_i) = 0, 1 \leqq i \leqq m\}$$

is an open neighborhood of zero in A.

We shall show that if $v \in V$, then $v(E) \subseteq M$. If not, let $v \in V$ and $y \in E$ be such that $v(y) \notin M$. Then $u + v \in G$, so rank $(u+v) \leqq m$. But $(u+v)(x_i) = u(x_i)$ if $1 \leqq i \leqq m$, and $(u+v)(y) = u(y) + v(y) \notin M$ since $v(y) \notin M$; hence $u(x_1), \ldots, u(x_m)$, $u(y) + v(y)$ is a linearly independent sequence of $m+1$ vectors belonging to the range of $u+v$, a contradiction.

Since E is infinite dimensional, there exist $y_1, \ldots, y_m \in E$ such that $u(x_1), \ldots, u(x_m), y_1, \ldots, y_m$ is a linearly independent sequence of $2m$ vectors. As A is dense, there exists $w \in A$ such that $w(u(x_i)) = y_i$ for $1 \leqq i \leqq m$. As $v \to wv$ is continuous, there is a neighborhood U of zero in A such that $U \subseteq V$ and $wU \subseteq V$. To show that $U = \{0\}$, let $v \in U$ and let $x \in E$. Then $v(x) \in M$, so there exist scalars $\lambda_1, \ldots, \lambda_m$ such that

$$v(x) = \sum_{i=1}^{m} \lambda_i u(x_i).$$

Consequently,

$$wv(x) = \sum_{i=1}^{m} \lambda_i y_i,$$

but $wv(x) \in M$ as $wv \in V$; hence $wv(x) = 0$, so $\lambda_1 = \cdots = \lambda_m = 0$, whence $v(x) = 0$. Thus $U = \{0\}$, so the topology of A is the discrete topology.

LEMMA 4. *Let E be a straight locally compact vector space over a discrete topological division ring K, and let A be a dense ring of continuous linear operators of finite rank on E, topologized so that A is a locally compact ring and $(u, x) \to u(x)$ is con-*

tinuous from $A \times E$ into E. If E is generated by a compact neighborhood V of zero, then A is discrete.

Proof. The ring A is a simple ring having minimal left ideals [6, Theorem 1]; let e be a minimal idempotent of A, i.e., a projection on a one-dimensional subspace M of E, and let N be the kernel of e. As E is straight, M is a discrete subspace and hence is closed, so $V \cap M$ is a compact discrete subset and hence is finite. Therefore there is an open neighborhood W of zero in E such that $W \cap M = \{0\}$. Now $N = e^{-1}(W)$, for if $x \in e^{-1}(W)$, then $e(x) \in W \cap M = \{0\}$, so $x \in N$; but $N = e^{-1}(W)$ is open as e is continuous. As $(u, x) \to u(x)$ is continuous, the topology of A is stronger than the compact-open topology on A [4, p. 46, Corollary 1]. Consequently,

$$U = \{u \in A : u(V) \subseteq N\}$$

is a neighborhood of zero in A. As V generates E,

$$U = \{u \in A : u(E) \subseteq N\},$$

so $eU = \{0\}$. Therefore eA is discrete.

As we observed earlier, A is anti-isomorphic as a topological ring to a dense ring A_R of linear operators of finite rank on the left eAe-vector space eA. As eA is discrete, A is not connected; as A is simple, therefore, A is totally disconnected, and thus the open additive subgroups of A form a fundamental system of neighborhoods of zero [2, p. 114, Exercise 18]. Also A is a Baire space as it is locally compact [4, p. 110, Theorem 1]. Thus by Lemma 3 applied to A_R, we conclude that A_R is discrete, whence A is also.

THEOREM 2. *A locally compact simple ring A having minimal left ideals is either discrete or isomorphic to the ring of all linear operators on a finite-dimensional vector space over a locally compact division ring.*

Proof. Let e be a minimal idempotent of A, let $K = eAe$, and let E be the right K-vector space Ae. As observed earlier, the latter conclusion follows if K is indiscrete; therefore we shall assume that K is discrete, and we shall prove that A is discrete. Let V be a compact neighborhood of zero in E, which we may assume contains a nonzero vector; then the subspace F generated by V is a nonzero subspace of E. Clearly F is a locally compact and hence closed subspace, so the subring

$$B = \{u \in A_L : u(F) \subseteq F\}$$

is a closed and hence locally compact subring of A_L. For each $u \in B$ let u_F be the restriction of u to F, and let $\rho: u \to u_F$. Then ρ is a homomorphism from B onto a subring B' of continuous linear operators of finite rank on F, and the kernel of ρ is the ideal

$$H = \{u \in B : u(F) = \{0\}\}.$$

Now H is clearly closed in B, so the topological ring B/H is Hausdorff and hence locally compact since the canonical epimorphism from B onto B/H is continuous and open. We topologize B' so that the algebraic isomorphism from B/H onto B' induced by ρ is a homeomorphism; then ρ is a continuous open epimorphism from B onto B'.

To show that B' is a dense ring of linear operators on F, let $x_1, \ldots, x_n$ be a sequence of linearly independent vectors of F and let $y_1, \ldots, y_n \in F$. Now A_L contains a projection p on the subspace generated by $\{y_1, \ldots, y_n\}$ [6, Lemma 1], and there exists $u \in A_L$ such that $u(x_i) = y_i$, $1 \leqq i \leqq n$. Then $pu(E) \subseteq F$, so the restriction v of pu to F belongs to B' and satisfies $v(x_i) = y_i$, $1 \leqq i \leqq n$. Moreover, F is straight since E is by Lemma 1, and $(v, x) \to v(x)$ is continuous from $B' \times F$ into F since $(u, x) \to u(x)$ is continuous from $A_L \times E$ into E and since ρ is an open mapping. Therefore by Lemma 4, B' is discrete. Consequently, H is open in B. But as V generates F,

$$B \supseteq \{u \in A_L : u(V) \subseteq V\},$$

which is a neighborhood of zero in A_L since the topology of A_L is stronger than the compact-open topology [4, p. 46, Corollary 1]. Hence B is an open subring of A_L, so H is an open left ideal of A_L.

For each $x \in E$, let

$$H_x = \{u \in A_L : u(x) = 0\}.$$

Let z be a nonzero vector of F. Then $H_z \supseteq H$, and hence H_z is an open left ideal of A_L. Let x be any vector of E, and let $g \in A_L$ be such that $g(z) = x$. Then there is a neighborhood W of zero in A_L such that $Wg \subseteq H_z$; hence $W \subseteq H_x$, so H_x is open. We now retopologize E with the discrete topology. Then for each $x \in E$, $u \to u(x)$ is continuous from A_L into E, since H_x is open in A_L. As K is discrete and thus not connected, A and hence A_L are not connected; therefore A_L is totally disconnected as it is simple, and consequently the open additive subgroups of A_L form a fundamental system of neighborhoods of zero [2, p. 114, Exercise 18]. Also A_L is a Baire space since it is locally compact [4, p. 110, Theorem 1]. Thus by Lemma 3, A_L is discrete; hence A is discrete.

2. **Locally compact vector spaces over discrete division rings.** Here we present a few results concerning straight locally compact vector spaces over discrete division rings, and from them we obtain a theorem on locally compact primitive algebras having minimal left ideals.

LEMMA 5. *If E is a straight Hausdorff topological vector space over a discrete division ring K and if V is a compact subset of E, then for every nonzero vector $x \in E$, $\{\lambda \in K : \lambda x \in V\}$ is finite.*

Proof. By hypothesis, $K.x$ is discrete and hence closed, so $V \cap K.x$ is compact and discrete and therefore finite; consequently $\{\lambda \in K : \lambda x \in V\}$ is finite.

THEOREM 3. *Let E be a straight locally compact vector space over a discrete infinite division ring K. If V is a compact neighborhood of zero in E and if* $(\lambda_n)_{n\geqq 1}$ *is any sequence of distinct nonzero scalars, then* $(\lambda_1 V \cap \cdots \cap \lambda_n V)_{n\geqq 1}$ *is a fundamental system of neighborhoods of zero in E; in particular, E is metrizable.*

Proof. For each $n \geqq 1$,

$$G_n = \{x \in V : \lambda_n^{-1}x \notin V\}$$

is an open subset of V, and $\bigcup_{n\geqq 1} G_n = V - \{0\}$ by Lemma 5. Let U be an open neighborhood of zero contained in V. Then $V-U$ is compact and is contained in $\bigcup_{n\geqq 1} G_n$, so for some $m \geqq 1$, $V - U \subseteq \bigcup_{k=1}^{m} G_k$. Therefore

$$U \supseteq V - \left(\bigcup_{k=1}^{m} G_k\right) = \bigcap_{k=1}^{m} (V - G_k) = \bigcap_{k=1}^{m} \{x \in V : \lambda_k^{-1}x \in V\}$$

$$= V \cap \lambda_1 V \cap \cdots \cap \lambda_m V.$$

Thus $(V \cap \lambda_1 V \cap \cdots \cap \lambda_n V)_{n\geqq 1}$ is a fundamental system of neighborhoods of zero. Replacing V by $\lambda_1 V$ and $\lambda_k \lambda_1^{-1}$, we conclude that $(\lambda_1 V \cap \lambda_2 V \cap \cdots \cap \lambda_n V)_{n\geqq 1}$ is a fundamental system of neighborhoods of zero.

THEOREM 4. *If E is a straight locally compact vector space over a discrete uncountably infinite division ring K, then E is discrete.*

Proof. Let V be a compact neighborhood of zero in E, let $(\lambda_n)_{n\geqq 1}$ be a sequence of distinct nonzero scalars, and let $V_m = \lambda_1 V \cap \cdots \cap \lambda_m V$ for each $m \geqq 1$. For each nonzero scalar μ, μV is a neighborhood of zero, and hence by Theorem 3 there exists $r(\mu) \geqq 1$ such that $V_{r(\mu)} \subseteq \mu V$. Since K is uncountably infinite, there exists $m \geqq 1$ such that $r(\mu) = m$ for infinitely many nonzero scalars μ. If x were a nonzero vector in V_m, then $x \in \mu V$ and hence $\mu^{-1}x \in V$ for infinitely many nonzero scalars μ, in contradiction to Lemma 5. Hence $V_m = \{0\}$, so E is discrete.

THEOREM 5. *If E is a straight locally compact vector space over a discrete division ring K whose characteristic is zero, then E is discrete.*

Proof. We assume that E is not discrete.

Case 1. There is a neighborhood of zero in E that contains no nonzero additive subgroup. By a theorem of Gleason [5, Lemma 1.4.2], there is a nonzero continuous homomorphism α from the topological additive group $\boldsymbol{R}$ of real numbers into the additive group E. Let $t_0 \in \boldsymbol{R}$ be such that $\alpha(t_0) \neq 0$, and let $x_0 = \alpha(t_0)$. Let β be the restriction of α to the subgroup of all rational multiples of t_0. As the characteristic of K is zero, β is injective, and its range is discrete as it is a subset of the one-dimensional subspace generated by x_0. Thus β is a continuous bijection from an indiscrete group onto a discrete group, which is impossible.

Case 2. Every neighborhood of zero in E contains a nonzero additive subgroup. Let x_0 be a nonzero member of an additive subgroup G contained in a compact neighborhood V of zero. Then $n.x_0 \in G \subseteq V$ for every integer n, so $\lambda x_0 \in V$ for infinitely many scalars λ as the characteristic of K is zero, in contradiction to Lemma 5.

It is easy to construct a straight, indiscrete, compact vector space over a discrete finite field: we need only consider the cartesian product of infinitely many copies of the field. By Theorems 4 and 5, there is no straight, indiscrete, locally compact vector space over a discrete division ring K if either K is uncountably infinite or K has characteristic zero. We next construct an example of a straight, indiscrete, locally compact vector space over a discrete, countable division ring of prime characteristic.

EXAMPLE 1. Let P be a prime field of prime characteristic, and let K be a countably infinite algebraic extension of P, e.g., let K be an algebraic closure of P. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration of the nonzero elements of K, let $K_0 = P$, and for each $n \geqq 1$ let $K_n = P[\lambda_1, \ldots, \lambda_n]$, the subfield generated by $\lambda_1, \ldots, \lambda_n$. As K is an algebraic extension of P, $(K_n)_{n \geqq 0}$ is an increasing sequence of finite subfields of K whose union is K. Let $E = K^N$, the K-vector space of all sequences $(\alpha_n)_{n \geqq 0}$ where $\alpha_n \in K$ for all $n \geqq 0$. Let $V = \prod_{n \geqq 0} K_n$, and for each $m \geqq 1$ let $V_m = \lambda_1 V \cap \cdots \cap \lambda_m V$. We shall prove that $(V_m)_{m \geqq 1}$ is a fundamental system of neighborhoods of zero for a topology on E making E an indiscrete, straight locally compact vector space over the discrete field K.

Since each V_m is an additive subgroup, addition is continuous on E. As K is discrete, to show that scalar multiplication is continuous it suffices to show that $x \to \lambda x$ is continuous at zero for each nonzero scalar λ. But if $m \geqq 1$, then $\lambda V_r \subseteq V_m$ where r is such that $\lambda^{-1}\lambda_1, \ldots, \lambda^{-1}\lambda_m$ are among $\lambda_1, \ldots, \lambda_r$. Thus E is a topological K-vector space.

To show that our topology is stronger than the cartesian product topology on $E = K^N$, let $W = \prod_{n \geqq 0} L_n$ where $L_n = \{0\}$ for $n < m$ and $L_n = K$ for $n \geqq m$. Let $\beta \in K$ be such that $\beta \notin K_m$; to show that $V \cap \beta V \subseteq W$, let $(\alpha_n)_{n \geqq 0} \in V \cap \beta V$. If $n < m$, then $\alpha_n \in K_n$ and $\beta^{-1}\alpha_n \in K_n$, so if $\alpha_n \neq 0$, then β^{-1} and hence also β would belong to $K_n \subseteq K_m$, a contradiction. Hence $\alpha_n = 0$ if $n < m$, so $(\alpha_n)_{n \geqq 0} \in W$, and thus $V \cap \beta V \subseteq W$. In particular, our topology is a Hausdorff topology.

Next we shall show that the topology induced on V is identical with the cartesian product topology of $V = \prod_{n \geqq 0} K_n$. By the preceding, it suffices to show that for each $m \geqq 1$ we have $H = \prod_{n \geqq 0} H_n \subseteq V_m$ where $H_n = \{0\}$ if $n < m$ and $H_n = K_n$ if $n \geqq m$. Let $(\alpha_n) \in H$, and let $n \geqq m$. Then $\alpha_n \in K_n$ and $\lambda_i^{-1} \in K_m \subseteq K_n$ if $1 \leqq i \leqq m$, so $\lambda_i^{-1}\alpha_n \in K_n$ if $1 \leqq i \leqq m$, and thus $\alpha_n \in \lambda_1 K_n \cap \cdots \cap \lambda_m K_n$. Therefore

$$(\alpha_n) \in \prod_{n \geqq 0} (\lambda_1 K_n \cap \cdots \cap \lambda_m K_n) = \lambda_1 V \cap \cdots \cap \lambda_m V = V_m.$$

In particular, our topology is indiscrete and V is compact for it; thus E is an indiscerte locally compact K-vector space.

It remains for us to show that E is straight, i.e., that every one-dimensional subspace of E is discrete. Let $z=(\alpha_n)_{n\geqq 0}$ be a nonzero vector, let $\alpha_s\neq 0$, and let $L=\{\lambda\in K : \lambda z\in V\}$. Then $L\alpha_s\subseteq K_s$, so L is finite since $\alpha_s\neq 0$ and K_s is finite. Therefore $K.z\cap V$ is finite, so $K.z$ is discrete.

To apply these results to locally compact primitive algebras having minimal left ideals, we first note that if A is a primitive algebra, then a minimal left ideal of the algebra A is also a minimal left ideal of the ring A since the right annihilator of A is the zero ideal. Therefore a primitive algebra having minimal left (algebra) ideals is also a primitive ring having minimal left ideals.

THEOREM 6. *Let A be a locally compact primitive algebra having minimal left ideals over a topological field K. If any one of the following three conditions holds, then either A is discrete or A contains an identity element and is finite-dimensional over its center.*

1° *K is indiscrete.*

2° *K has characteristic zero.*

3° *K is uncountably infinite.*

Proof. Let e be a minimal idempotent of A, and let $D=eAe$.

Case 1. D is indiscrete. Then D is an indiscrete locally compact division ring and hence is finite dimensional over its center [8, Theorem 8], and as we observed earlier A is isomorphic to the ring of all linear operators on a finite-dimensional D-vector space. Hence A has an identity element and the center of D may be identified with the center of A, so A is finite dimensional over its center.

Case 2. D is discrete. As $\lambda\to\lambda e$ is a continuous injection from K into D, K is also discrete and hence by our hypothesis either has characteristic zero or is uncountably infinite. To show that the K-vector space A is straight, let a be a nonzero element of A, and let $x\in Ae$ be such that $a_L(x)\neq 0$. As we saw earlier, $v\to v(x)$ is continuous from A_L into Ae, and its restriction to $(K.a)_L$ is an injection from $(K.a)_L$ into $a_L(x).D$, since

$$(\lambda a)_L(x) = (\lambda a)x = (\lambda a)(xe) = (ax)(\lambda e)\in a_L(x).D$$

as $x=xe$. By Lemma 1, $a_L(x).D$ is discrete, so $K.a$ is also discrete. Therefore A is straight, so by Theorems 4 and 5, A is discrete.

COROLLARY. *If A is a locally compact central primitive algebra having minimal left ideals over a topological field K and if K either is indiscrete, has characteristic zero, or is uncountably infinite, then A is finite dimensional over K.*

Proof. If A is indiscrete, then by Theorem 6 A has an identity element 1, and hence $\lambda\to\lambda.1$ is a bijection from K onto the center of A as A is central; therefore A is finite dimensional over K by Theorem 6.

Kaplansky [9, p. 458] constructed for any finite field K an indiscrete locally compact primitive K-algebra that has an identity element and minimal left ideals but is infinite dimensional over its center. We construct another example of such an algebra where K is countably infinite and has prime characteristic.

EXAMPLE 2. We continue with the notation of Example 1. Let F be the subspace of E generated by V. Then F is locally compact, metrizable, and

$$F = \bigcup_{n \geqq 1} (\lambda_1 V + \cdots + \lambda_n V),$$

the union of a countable sequence of compact open subgroups; hence F is also separable [3, p. 43, Corollary]. Consequently $\mathscr{C}(F)$, the K-algebra of all continuous functions from F into F, equipped with the compact-open topology, is a separable, metrizable, topological K-algebra [4, p. 47, Proposition 9; p. 34, Corollary; p. 41, Corollary]. We shall show that the subalgebra A of all continuous linear operators on F is indiscrete and locally compact. Clearly A is a closed subalgebra of $\mathscr{C}(F)$. Let

$$U = \{u \in A : u(V) \subseteq V\}.$$

Then U is a neighborhood of zero in A, and U is clearly closed in A and hence in $\mathscr{C}(F)$. Also, U is equicontinuous, for $u(V_m) \subseteq V_m$ for all $u \in U$ and all $m \geqq 1$. To show that U is compact, therefore, it suffices by Ascoli's Theorem [4, p. 32, Corollary 3] to show that $U(x)$ is relatively compact for each $x \in F$. But if $x \in F$, then $x \in \lambda_1 V + \cdots + \lambda_m V$ for some $m \geqq 1$, whence $u(x) \in \lambda_1 V + \cdots + \lambda_m V$ for all $u \in U$; thus $U(x) \subseteq \lambda_1 V + \cdots + \lambda_m V$, a compact set. Hence A is a locally compact, separable, metrizable K-algebra.

For each $m \geqq 0$ let e_m be the projection defined by

$$e_m((\alpha_n)) = (\delta_{nm}\alpha_n)$$

for all $(\alpha_n) \in F$, where δ_{nm} is the Kronecker notation. Then $e_m^{-1}(V) \supseteq V$, so $e_m^{-1}(V_r) \supseteq V_r$ for all $r \geqq 1$, and hence $e_m \in A$. Thus A contains nonzero linear operators of finite rank. To show that A is dense, it suffices to show that A is 2-fold transitive. Let $\{(\alpha_n), (\beta_n)\}$ be a linearly independent set of two vectors of F. If $\alpha_m = 0$ and $\beta_m \neq 0$ for some $m \geqq 0$, then $e_m((\alpha_n)) = 0$ and $e_m((\beta_n)) \neq 0$; otherwise $\alpha_m = 0$ implies that $\beta_m = 0$ for all $m \geqq 0$, so there exist r, s such that $\alpha_r \neq 0$, $\alpha_s \neq 0$, and $\beta_r/\alpha_r \neq \beta_s/\alpha_s$, whence $u = \alpha_r^{-1}e_r - \alpha_s^{-1}e_s$ satisfies $u((\alpha_n)) = 0$, $u((\beta_n)) \neq 0$. Since A contains the identity linear operator 1, $\lambda \to \lambda.1$ is therefore an isomorphism from K onto the center of A. Therefore A is an infinite-dimensional central primitive K-algebra having minimal left ideals and an identity element.

It remains for us to show that A is indiscrete; we shall show that $e_m \to 0$. Every compact subset of F is contained in $\lambda_1 V + \cdots + \lambda_n V$ for some $n \geqq 1$. If $m \geqq n$, then $\lambda_1, \ldots, \lambda_n \in K_m$, and hence $e_m(\lambda_1 V + \cdots + \lambda_n V) \subseteq \lambda_1 V \cap \cdots \cap \lambda_n V$. Therefore $e_m \to 0$.

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